Inequality with ratio of two nested quadratic roots.
https://www.linkedin.com/feed/update/urn:li:activity:6550652256092598272
Proving Inequality

$$\frac{2-\overbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+\sqrt{2}}}}}^{n-1\ roots}}{2-\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+\sqrt{2}}}}}_{n-1\ roots}} > \frac{1}{4}$$

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Let $a_n := \sqrt{2+\sqrt{2+\ldots+\sqrt{2+\sqrt{2}}}}$ (n square roots), $n \in \mathbb{N}$. Then $a_1 = \sqrt{2}$,

$a_{n+1} = \sqrt{2+a_n}, n \in \mathbb{N}$ and $\dfrac{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2+\sqrt{2}}}}}{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2+\sqrt{2}}}}} = \dfrac{2-a_{n+1}}{2-a_n}$.

Note that $\dfrac{2-a_{n+1}}{2-a_n} = \dfrac{4-a_{n+1}^2}{(2-a_n)(2+a_{n+1})} = \dfrac{4-(2+a_n)}{(2-a_n)(2+a_{n+1})} = \dfrac{1}{2+a_{n+1}}$.

We will find upper bound for $a_n, n \in \mathbb{N}$ by the following way. Assuming that $M > 0$ is upper bound for $a_n, n \in \mathbb{N}$ that is $a_n \le M$ for any $n \in \mathbb{N}$ we claim

$$\sqrt{2+M} \le M \Leftrightarrow 0 \le M^2 - M - 2 \overset{M>0}{\Leftrightarrow} M \ge 2.$$

Let $M := 2$. Then by Math Induction $a_n \le 2$ for any $n \in \mathbb{N}$, since $a_1 < 2$ and for any $n \in \mathbb{N}$ we have $a_n < 2 \Rightarrow a_{n+1} = \sqrt{2+a_n} < \sqrt{2+2} = 2$.

Thus, $\dfrac{1}{2+a_{n+1}} > \dfrac{1}{2+2} = \dfrac{1}{4}$.